Lesson: The Real Number System

Eighth Grade Objective: 1.01 Develop number sense for the real numbers.

a. Define and use irrational numbers.
b. Compare and order.
c. Use estimates of irrational numbers in appropriate situations.

Lesson:

The Real Number System includes any and all numbers that fit into the following categories:

1. Natural Numbers (Counting Numbers): These are the numbers that you would use if you were playing hide and go seek with a three year old: 1, 2, 3, 4, …
2. Whole Numbers: These are the Natural Numbers and zero: 0, 1, 2, 3, 4, …
3. Integers: These are Whole Numbers and their opposites: …-3, -2, -1, 0, 1, 2, 3…
4. Rational Numbers: These are any numbers that can be written as a fraction, a/b, where a and b are both integers and b is not equal to zero.
5. Irrational Numbers: These numbers cannot be written as a fraction, for example, non-repeating, non-terminating decimals.

Irrational numbers are those that cannot be written in the form a/b; where a and b are both integers and b is not equal to zero. Examples of irrational numbers are pi, e, and ð2. We can use estimates of those numbers to solve problems, but if we choose to use rational estimates for irrational numbers, our answers will not be exact. Instead, we can compute mathematically and simplify using properties of irrational numbers.

Examples:
Find the exact area of a circle with diameter 8 inches.
The area of a circle can be computed using the formula $a = \prod r^2$. Instead of using 3.14 or 22/7 to represent ∏, we are going to use ∏ as a number itself.

$$a = \prod r^2$$
$$a = \prod 4^2$$
$$a = 16\prod \text{ square inches}$$

This is an exact answer. Had we chosen to use 3.14 or 22/7, we would have arrived at an estimate close to the exact value, but an estimate still.

One circle is drawn inside another. The larger circle has radius 5 and the smaller circle has radius 2. What is the difference in the areas of the two circles?
The area of the larger circle is $a = \prod 5^2$, or $25\prod$.
The area of the smaller circle is $a = \prod 2^2$, or $4\prod$.
At this point, we can treat ∏ as though it were a variable, $25\prod - 4\prod = 21\prod$.

One circle is drawn inside another. The larger circle has radius 9 and the smaller circle has radius 5. A dart is thrown at the circles. What is the probability of the dart landing inside the larger circle, but outside the smaller circle?

The area of the larger circle is a = ∏9^2, or 81∏.
The area of the smaller circle is a = ∏5^2, or 25∏.
At this point, we can treat ∏ as though it were a variable, 81∏ - 25∏ = 56∏.
To find the probability, we put the difference in the areas over the area of the larger circle: 56∏/81∏ and simplify. 56/81 does not simplify, but ∏/∏ simplifies to 1.
Therefore the probability of the dart landing inside the larger circle, but outside the smaller circle is 56/81.

We already know that some radicals can be simplified exactly: ð9, for example, is 3. Can ð45 be simplified exactly? Yes, it can, because a perfect square divides, without remainder, into it. To simplify non-perfect square roots, make a mental or written list of perfect squares: 1, 4, 9, 16, 25, 36, 49, 64, ….
Determine if any perfect square divides into the number without remainder, begin with the largest perfect square that is ½ the value of the number, since any number larger will not divide into the number without a remainder.
So think: ½ of 45 = 22.5. The largest perfect square that is smaller than 22.5 is 16. Does 16 go into 45: no. Try the next smaller number. Does 9 go into 45: yes.
Rewrite your radical as a product of a number and a perfect square: ð45 = ð(9*5) = ð9 * ð5 = 3ð5.
This is an exact value for ð45, had we typed this into the calculator, we would have arrived at a decimal approximation. Close, but not exact.

Simplify:

1. ð90
   Half of the number is 45, the largest perfect square that is less than 45 is 36, which does not divide evenly into 90. The next largest is 25, which also does not divide evenly. Neither does 16. But 9 does, so we can rewrite the radical as ð90 = ð(9*10) = ð9*ð10=3ð10
2. ð120
   Half of the number is 60, the largest perfect square that is less than 60 is 49, which does not divide evenly into 120. The next largest is 36, which also does not divide evenly. Neither does 25, nor 16, nor 9. But 4 works, so we can rewrite the radical as ð120 = ð(4*30) = ð4 * ð30 = 2ð30.
3. ð128
   Sometimes, a perfect square might just "jump out" at you. In this example, I notice immediately that 4 goes into 128. I rewrite ð128 as ð(4*32) = ð4 * ð32 = 2ð32. I am completely correct, so far, but I am not finished simplifying. It is possible to continue to simplify ð32. 2ð32 can be written as 2ð(16*2) = 2ð16 *ð2 = 2*4ð2 or 8ð2. Don't forget to check to make sure that your final answer is in simplest radical form – meaning that no perfect squares can be "taken out" of your final radical.

Application:

1. What is the exact length of the hypotenuse of a right triangle whose legs are 12 cm and 8 cm?

Using the Pythagorean theorem, we begin with $12^2 + 8^2 = c^2$

$144 + 64 = c^2$

$208 = c^2$

To undo the square, we need to take the square root of 208. Half of the number is 104, the largest perfect square that is less than 104 is 100, which does not divide evenly into 208. The next largest is 81, which also does not divide evenly. Neither does 64, nor 49, nor 36, nor 25. But 16 works, so we can rewrite the radical as ð208 = ð(16*13) = ð16 * ð13 = 4ð13. The hypotenuse is 4ð13 cm long.

2. What is the exact length of the legs of an isosceles right triangle whose hypotenuse measures 14 meters?

Using the Pythagorean theorem, we begin with $a^2 + a^2 = 14^2$

$2a^2 = 196$

$a^2 = 98$

a = ð98

Half of 98 is 49, which is perfect, so we can rewrite ð98 as ð(49*2) = ð49 * ð2 = 7ð2. Both legs of the isosceles right triangle are 7ð2.

3. Find the perimeter of the triangle in question 2.

Each leg is 7ð2 and the hypotenuse is 14. That leaves us the equation p = 7ð2 + 7ð2 + 14. You can add and subtract radical expressions as long as they have the same number/variables under the radical sign. You treat the radicals more or less as variables: p = 7ð2 + 7ð2 + 14

p = 14ð2 + 14

p = 14 + 14ð2

Notice that you do not add the numbers under the radical sign!

Try these on your own:

1. Find the exact area of a circle with radius 7 ft.
2. A circle with radius 3 is drawn inside a larger circle with radius 10. A dart is randomly thrown at the circles. What is the probability that the dart lands within the larger circle, but outside the smaller circle?
3. Simplify: ð150
4. Simplify: ð539
5. What is the exact perimeter of a right triangle whose legs measure 4 and 7 units?

Check your answers:

1. The area of a circle can be computed using the formula a = ∏r^2.

a = ∏r^2

a = ∏7^2

a = 49∏ square inches

2. The area of the larger circle is a = ∏10^2, or 100∏.

The area of the smaller circle is a = ∏3^2, or 9∏.

At this point, we can treat ∏ as though it were a variable, 100∏ - 9∏ = 91∏.

To find the probability, we put the difference in the areas over the area of the larger circle: 91∏/100∏ and simplify. 91/100 does not simplify, but ∏/∏ simplifies to 1. Therefore the probability of the dart landing inside the larger circle, but outside the smaller circle is 91/100.

3. 150/2 = 75. The largest perfect square that is less that 75 is 64, which does not go into 150. The next is 49, which also does not divide evenly. The next is 36, which does not divide evenly. The next is 25, which does, therefore: ð150 = ð(25*6) = ð25 * ð6 = 5ð6.

4. 539/2 = 269.5. The largest perfect square smaller than 269.5 is 256, which does not divide evenly into 539. Neither does: 225, 196, 169, 144, 121, 100, 81, or 64. 49 does. Therefore: ð539 = ð(49*11) = ð49 * ð11 = 7ð11.

5. Using the Pythagorean theorem: $4^2 + 7^2 = c^2$

$$16 + 49 = c^2$$

$$65 = c^2$$

ð65 is the length of the hypotenuse and it cannot be simplified (nothing perfect divides evenly into it). Our perimeter is 4 + 7 + ð65 or 11 + ð65.

Quiz Yourself:

1. Two smaller circles, both with radius 2, are drawn inside a circle with radius 9. A dart is randomly thrown at the larger circle. What is the probability that the dart will land within the larger circle, but outside the two smaller circles?
2. Simplify: ð252
3. What is the perimeter of an isosceles right triangle whose hypotenuse is 10 cm long?

Check your answers:

1. 73/81 or about 90%
2. 6ð7
3. 10 + 10ð2